

A note on multiple Seshadri constants on surfaces.

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Abstract

We give a bound for the multiple Seshadri constants on surfaces with Picard number 1. The result is a natural extension of the bound of A. Steffens for simple Seshadri constants. In particular, we prove that the Seshadri constant $\epsilon(L; r)$ is maximal when rL^2 is a square.

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1 Introduction.

The multiple Seshadri constants are a natural generalization of the Seshadri constants at single points defined by Demailly in [1]. If X is a smooth projective surface, L is a nef bundle on X and P_1, \dots, P_r are distinct points in X , then the Seshadri constant of L at P_1, \dots, P_r is:

$$\epsilon(L; P_1, \dots, P_r) = \inf \frac{C \cdot L}{\sum_{i=1}^r \text{mult}_{P_i} C}$$

where C runs over all curves passing through at least one of the points P_1, \dots, P_r . When the points are general we will write $\epsilon(L; r)$. These constants have the upper bound:

$$\epsilon(L; r) \leq \sqrt{\frac{L^2}{r}}.$$

However, explicit values are difficult to compute even when $r = 1$. In [4], A. Steffens proved the following result for simple Seshadri constants on surfaces with Picard number 1:

Proposition 1.1 *Let X be a surface with $\rho(X) = \text{rk}(NS(X)) = 1$ and let L be an ample generator of $NS(X)$. Let α be an integer with $\alpha^2 \leq L^2$. If $x \in X$ is a very general point, then $\epsilon(L, x) \leq \alpha$. In particular, if $\sqrt{L^2}$ is an integer, then $\epsilon(L, x) = \sqrt{L^2}$.*

Some results in the same direction have been proved for multiple Seshadri constants. In [2], Harbourne defines:

$$\varepsilon_{r,l} = \max \left\{ \frac{\lfloor d\sqrt{rl} \rfloor}{dr}, \quad 1 \leq d \leq \sqrt{\frac{r}{l}} \right\} \cup \left\{ \frac{1}{\lceil \sqrt{\frac{r}{l}} \rceil} \right\} \cup \left\{ \frac{dl}{\lceil d\sqrt{rl} \rceil}, \quad 1 \leq d \leq \sqrt{\frac{r}{l}} \right\}$$

and he shows the following bound:

Theorem 1.2 *Let $l = L^2$, where L is a very ample divisor on an algebraic surface X . Then $\epsilon(L; r) \geq \varepsilon_{l,r}$, unless $l \leq r$ and rl is a square, in which case $\sqrt{l}, r = \varepsilon_{r,l}$ and $\epsilon(L; r) = \sqrt{l}/r$.*

When $l \leq r$ and L is very ample this implies:

$$\epsilon(L; r) \geq \frac{\lfloor \sqrt{rL^2} \rfloor}{r}.$$

Moreover, if rL^2 is a square, $\epsilon(l; r)$ is maximal.

On the other hand, J. Roe in [3] relates the simple and multiple Seshadri constants. As a consequence of his main theorem and the result of Steffens, he obtains:

Corollary 1.3 *Let X be a smooth projective surface defined over \mathbb{C} , L an ample generator of $NS(X)$ and $r \geq 9$. The Nagata's conjecture implies:*

$$\epsilon(L, r) \geq \frac{\lfloor \sqrt{L^2} \rfloor}{\sqrt{r}}.$$

In this note, we extend the result of Steffens for multiple Seshadri constants. We prove:

Theorem 1.4 *Let X be a surface with $\rho(X) = rk(NS(X)) = 1$ and let L be an ample generator of $NS(X)$. Then*

$$\epsilon(L; r) \geq \frac{\lfloor \sqrt{rL^2} \rfloor}{r}.$$

In particular, if rL^2 is a square, $\epsilon(L; r)$ is maximal.

In this case, the Harbourne's hypothesis of very ampleness of L and $L^2 \leq r$ are not necessary. Furthermore, we do not need to use the Nagata's conjecture.

The proof of the theorem is a natural generalization of the method of Steffens.

2 Proof of the Theorem.

Let X be a surface with $\rho(X) = 1$ and let L be an ample generator of $NS(X)$. Let α be an integer with $\alpha^2 < rL^2$. Let us suppose that:

$$\epsilon(L; r) < \frac{\alpha}{r} \leq \sqrt{\frac{L^2}{r}}.$$

Then, there is a Seshadri exceptional curve C with multiplicities (m_1, \dots, m_r) at very general points, such that:

$$\epsilon(L, r) = \frac{C \cdot L}{M}, \quad \text{where} \quad M = \sum_{i=1}^r m_i.$$

In order to bound this multiplicities, let us recall two lemmas.

Lemma 2.1 *Let X be a smooth surface and let $(C_t, (P_1)_t, \dots, (P_r)_t)$ be a one parameter family of pointed curves on X with $\text{mult}_{(P_i)_t}(C_t) = m_i$. Then:*

$$C_t^2 \geq \sum_{i=1}^r m_i^2 - \min(m_1, \dots, m_r).$$

Proof: See [6]. ■

Lemma 2.2 *Let (X, L) be a polarized surface with Picard number 1 and let P_1, \dots, P_r be general points on X . If $\epsilon(L; P_1, \dots, P_r) < \sqrt{L^2/r}$ then any irreducible Seshadri curve is almost-homogeneous.*

Proof: See [5]. ■

Corollary 2.3 *With the previous notation:*

$$rC^2 \geq M(M-1).$$

Proof: Applying the two lemmas, we know that the multiplicities of C are $(m_1, \dots, m_r) = (a, \dots, a, b)$ and

$$rC^2 \geq r(r-1)a^2 + rb^2 - r \min(a, b).$$

From this:

$$\begin{aligned} rC^2 - M(M-1) &\geq \\ &\geq r(r-1)a^2 + rb^2 - r \min(a, b) - ((r-1)a + b)^2 + (r-1)a + b = \\ &= (r-1)a^2 + (r-1)b^2 - 2ab(r-1) + (r-1)a + b - r \min(a, b) = \\ &= (r-1)(a-b)^2 + (r-1)a + b - r \min(a, b). \end{aligned}$$

If $a \geq b$ then it holds:

$$rC^2 - M(M-1) \geq (r-1)((a-b)^2 - (a-b)) \geq 0.$$

When $a < b$:

$$rC^2 - M(M-1) \geq (r-1)(a-b)^2 + b-a \geq 0.$$

■

Now, we can extend the proof of Steffens. We have:

$$\epsilon(L; r) < \frac{\alpha}{r} \Rightarrow \frac{C \cdot L}{M} < \frac{\alpha}{r} \Rightarrow rC \cdot L < \alpha M.$$

On the other hand, since $\rho(X) = 1$, there is an integer d such that $C \equiv dL$ and:

$$rdL^2 < \alpha M \Rightarrow \alpha^2 d < \alpha M \Rightarrow \alpha d < M \Rightarrow \alpha d \leq M-1.$$

Thus, applying the bound of the previous Corollary:

$$M(M-1) \leq rC^2 = rdC \cdot L < \alpha dM \leq M(M-1),$$

and this is a contradiction. ■

References

- [1] DEMAILLY, J.-P. *Singular hermitian metrics on positive line bundles* Lecture Notes Math. **1507**, 87-104 (1992).
- [2] HARBOURNE, B. *Seshadri constants and very ample divisors on algebraic surfaces*. J. Reine Angew. Math **559**, 115-122 (2003).
- [3] ROE, J. *A relation between one-point and multi-point Seshadri constants*. J. Algebra **274**, N2, 643-651 (2004).
- [4] STEFFENS, A. *Remarks on Seshadri constants*. Math. Z. **227**, 505-510 (1998).
- [5] STRYCHARZ-SZEMBERG, B.; SZEMBERG, T. *Remarks on the Nagata conjecture*. Serdica Math. J. **30**, N 2-3, 405-430 (2004).
- [6] XU, G. *Ample line bundles on smooth surfaces*. J. Reine Angew. Math. **469**, 199-209 (1995).

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